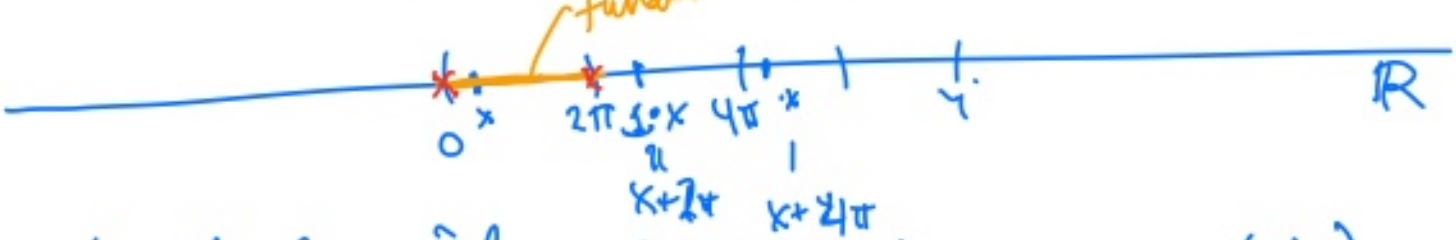
Series of talks – R-torsion.

$\mathbb{R}$ 1-dim'l manifold

<u>Action:</u> $\mathbb{Z}$ acts on $\mathbb{R}$

Given $x \in \mathbb{R}$, $m \in \mathbb{Z}$ $\quad m \cdot x = x + m2\pi$



The set of equivalence classes (i.e. orbits of this action) as a set. $[x] = \{x+2\pi, x-2\pi, x+4\pi, x-4\pi, \dots\}$

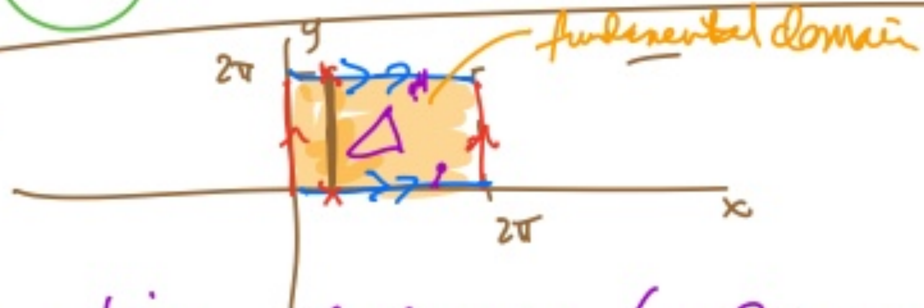
$\Rightarrow \mathbb{R}/\mathbb{Z} = \{[x]\} =$

$\shortparallel$

S^1

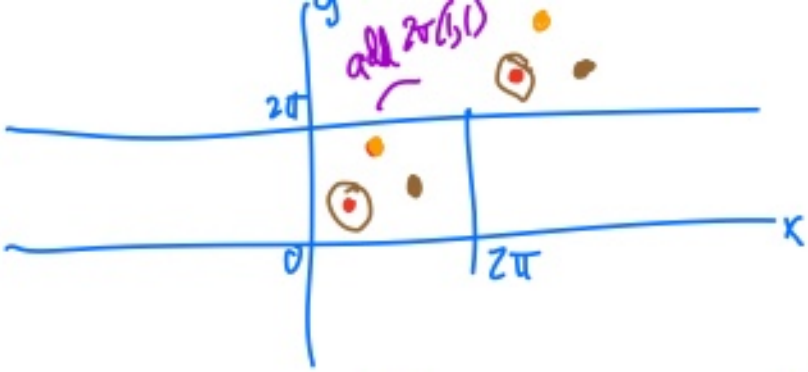
Circle

$T^2 = \mathbb{R}^2/_{2\pi\mathbb{Z}^2}$

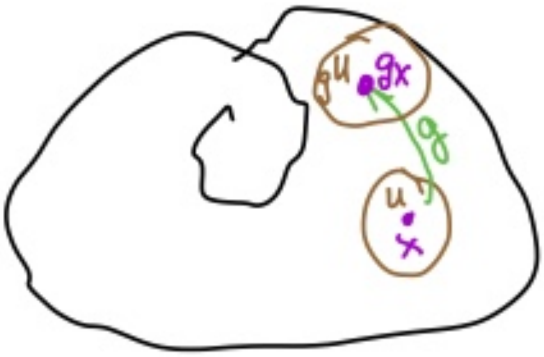


group action $\quad (x,y) \rightsquigarrow (x+2\pi m, y+2\pi q)$ where $m, q \in \mathbb{Z}$.

Both of these group actions on $\mathbb{R}$ & $\mathbb{R}^2$ are actions by isometries – length & angle-preserving.



These group actions are **properly discontinuous actions**. G acting on M



$$\boxed{\forall x \in M,\ \exists \text{ nbhd } U \text{ containing } x \text{ such that } \forall g \in G \quad U \cap gU = \emptyset.}$$

Thm If M is a connected & simply connected manifold & G acts on M properly discontinuously, then

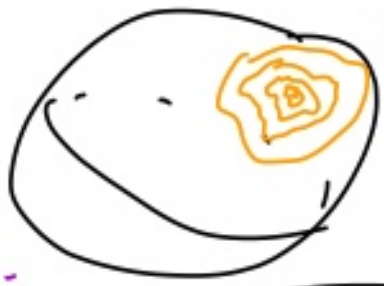
$$\pi_1(M/G) = \text{"fundamental group of } M/G\text{"} \cong G$$

This is a topological invariant.

$\mathbb{R}$ & $\mathbb{R}^2$ & $\mathbb{R}^4$ are simply connected.

simply connected — every closed curve on the space can be continuously contracted to a point.

simply connected.

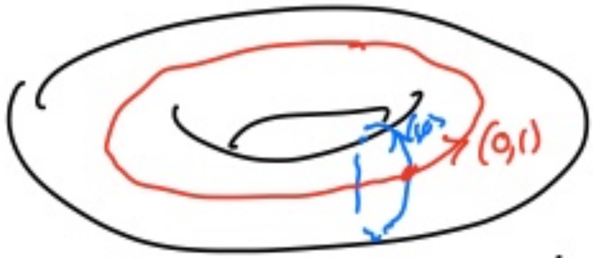


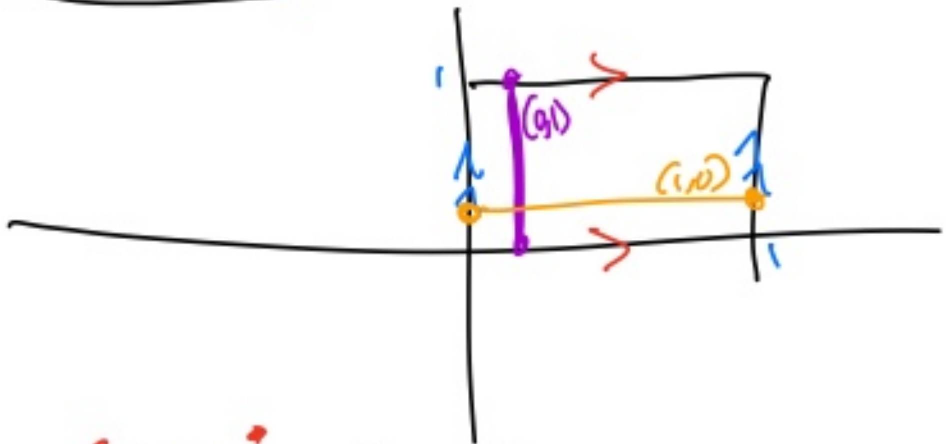
Not simply connected

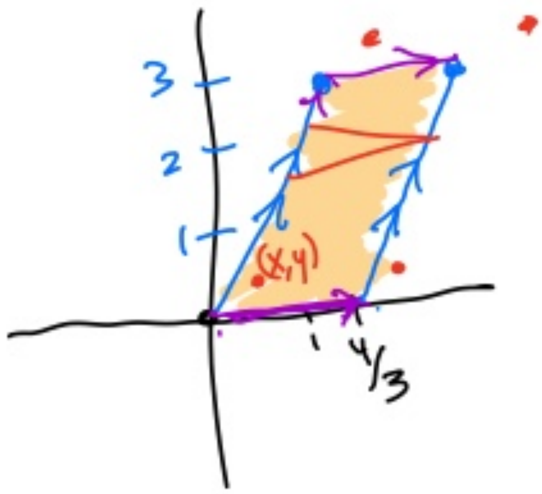
can't be contracted to a point

$T^2 = \underset{\uparrow \atop \text{s.conn.}}{\mathbb{R}^2} / 2\pi\mathbb{Z}^2 \Rightarrow \pi_1(T^2) \cong \mathbb{Z}^2 \cong 2\pi\mathbb{Z}^2.$

generators
$(1,0)$
$(0,1)$





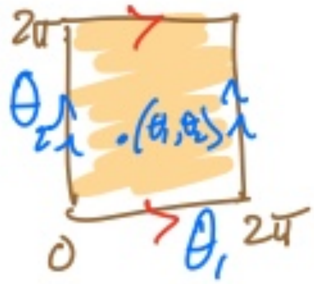


$\mathbb{R}^2 / G$ a torus.

G = group generated by

$\underline{(1,3)}$, $\underline{(\tfrac{4}{3}, 0)}$ (vector addition)

$(x,y) + m(1,3) + n(\tfrac{4}{3}, 0)$

Embedding 

into $\mathbb{R}^4$.

$$F(\theta_1,\theta_2) = (\underbrace{e^{i\theta_1}}_{\text{circle}}, \underbrace{e^{i\theta_2}}_{\text{circle}}) \in \mathbb{C}^2$$

$$= \underline{(\cos\theta_1, \sin\theta_1, \cos\theta_2, \sin\theta_2)} \in \mathbb{R}^4$$

$$S^3 = \{ (x_1,x_2,x_3,x_4) \in \mathbb{R}^4 : x_1^2+x_2^2+x_3^2+x_4^2 = 1 \}$$

$$= \{ (z_1,z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1 \}.$$

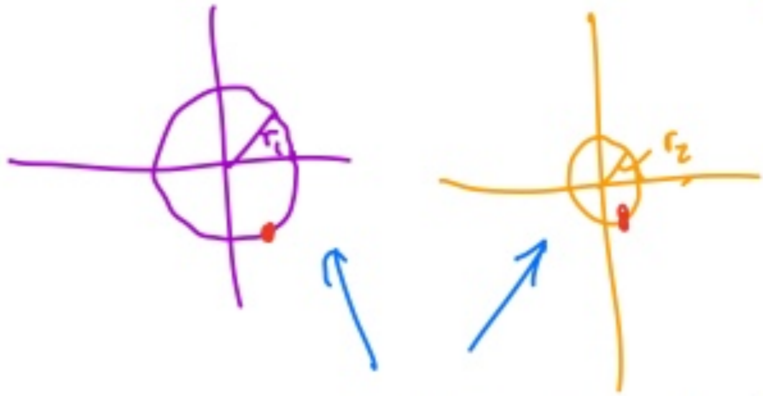
$$= \{ q \in \mathbb{H} : |q| = 1. \}$$

How to think of S^3

$$S^3 = \mathbb{R}^3 \cup \{\infty\}$$

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : \underbrace{|z_1|}_{r_1}{}^2 + \underbrace{|z_2|}_{r_2}{}^2 = 1\}$$

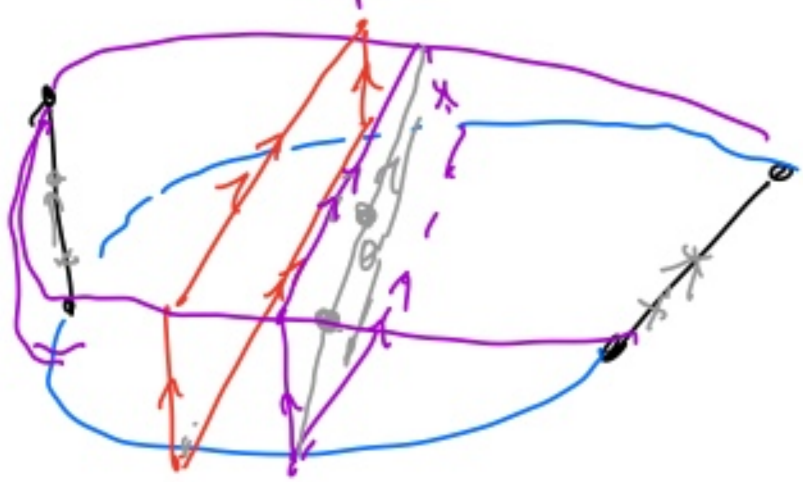
$r_1^2 + r_2^2 = 1$



a T^2 inside $\mathbb{C}^2$.

S^3 = union of tori (radii (r_1, r_2) on unit circle.

(and 2 circles $r_1 = 0, r_2 = 1$
$r_1 = 1, r_2 = 0$)



Another way to think of S^3.

A group action on S^3.

Given (z_1, z_2) on S^3 $\quad (|z_1|^2 + |z_2|^2 = 1)$

Given $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, let

$$\phi(\theta)(z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2)$$

Notice

$$\phi : \underset{\underset{\{\theta\}}{S^1 =}}{\overset{\shortparallel}{G}} \to \begin{pmatrix}\text{isometries} \\ \text{of } \mathbb{C}^2\end{pmatrix}.$$

$$|e^{i\theta} z_1| = |e^{i\theta}|\,|z_1| = |z_1|$$

$$|e^{i\theta} z_2| = |z_2|.$$

It maps the torus inside S^3 to itself. $\longrightarrow$ maps S^3 to itself.

Every "orbit" of this group action is a circle.

Quotient $\quad S^3 / S^1 \cong S^2$

$\uparrow$ can prove this in the same.

Called the Hopf Fibration.

Finite group action on S^3

$\mathbb{Z}_p = \{\text{integers mod } p\}$ $(p \in \mathbb{N})$

Let $q \in \mathbb{N}$ s.t. $\gcd(p, q) = 1$

q is relatively prime to p

The Lens space is

$$S^3/\mathbb{Z}_p = \{(z_1, z_2) \in S^3\}/\text{action}.$$

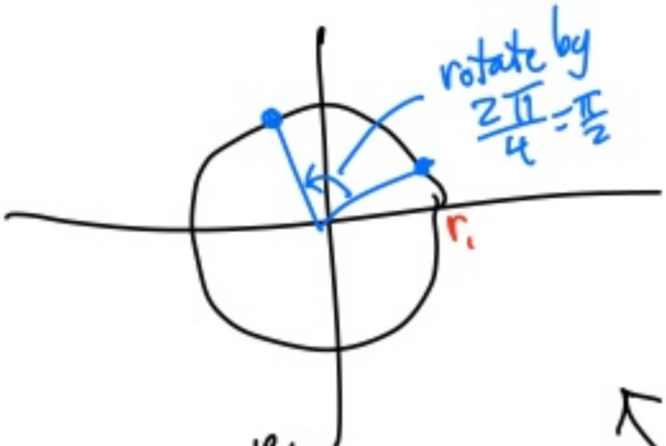
This is a properly discontinuous action.

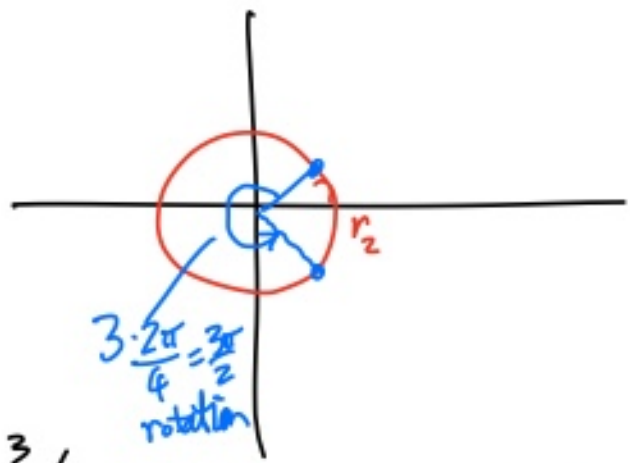
Take $a \in \mathbb{Z}_p$

$$a \cdot (z_1, z_2) = \left(e^{i\frac{2\pi a}{p}} z_1, e^{i\frac{2\pi q a}{p}} z_2\right)$$

(This is actually an isometric action.)

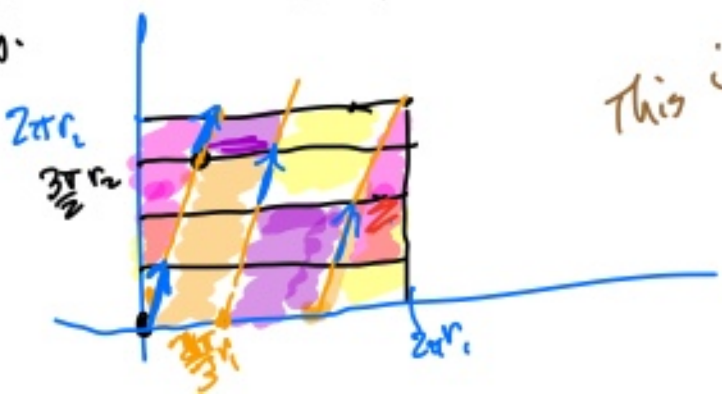
Example: $p=4$ $q=3$





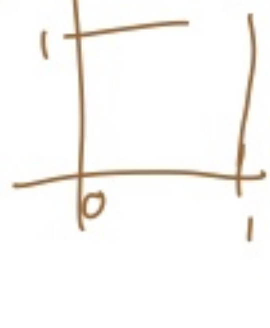
$L(4,3) = S^3/\mathbb{Z}_4$

What does this look like on each torus.



This is like

$T^2 = S^1 \times S^1$ mod $\left(\frac{1}{4}\ \frac{3}{4}\right)$



Big Questions about $L(p,q)$:

Defined by Tietze 1908 "torus manifolds"

"lens spaces" Seifert & Threlfall.

$$\pi_1(L(p,q)) = \pi_1\left(S^3/\mathbb{Z}_p\right) \cong \mathbb{Z}_p$$

fundamental group of $L(p,q)$ since S^3 is simply connected & connected

An old open question: Is a oriented 3-dim closed manifold determined by its fundamental group?

[Tietze proved in 1908 that all previously defined invariants of 3-manifolds are determined by π_1.)

Tietze guessed that the conjecture is false, and he suspected that the $L(p,q)$'s for different q-values would provide counterexamples.

In 1919, JW Alexander proved $L(5,1) \not\cong L(5,2)$

even though they have same π_1, $H_*(M)$.

His argument could actually show that

$$\text{If } \underset{\substack{\uparrow\\ \text{homeomorphic}\\ \text{"topologically same"}}}{L(p,q) \cong L(p,q')} \Rightarrow qq' = \pm r^2 \bmod p$$

for some $r \in \mathbb{Z}_p$.

In 1941, Whitehead prove that $L(p,q) \simeq L(p,q')$ (is homotopy equivalent)

$\Leftrightarrow qq' = \pm r^2 \bmod p \Rightarrow L(5,1) \not\simeq L(5,2)$

Reidemeister (1935) showed

$L(p,q) \cong L(p,q') \Leftrightarrow q = \pm q' \bmod p$ or $qq' = \pm 1 \bmod p$.

homeomorphic

using "Reidemeister torsion"

$\Rightarrow L(7,1) \simeq L(7,2)$ but $L(7,1) \not\cong L(7,2)$.

whitehead homotopy equ.

$\therefore\ \exists$ closed, or. 3 manifolds that are homot. equivalent but not homeomorphic

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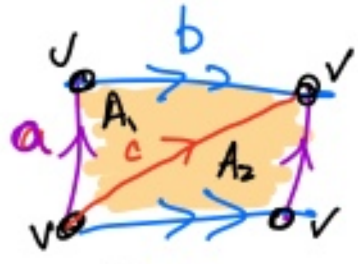
1952 Moise — Proved Haupt vermutung in dim. 3

<u>Thm</u> - Every (closed, oriented) 3-mfld has a unique PL structure and a unique smooth structure.

<u>1961 Milnor</u>: Hauptvermutung (PL version) is false in dim ≥ 4

<u>Homology</u> — take your space / manifold — decompose in cells/polyhedra.



set of 0 chains $=$ $C_0 = \text{span}_{\mathbb{Z}}\{0\text{-cells}\} = \text{span}\{\text{vertices}\}$
$= \{Av\}_{A \in \mathbb{Z}}$

set of 1-chains $=$ $C_1 = \text{span}\{a, b, c\}$

$C_2 = \text{span}\{A_1, A_2\}$

Boundary map ∂

$$C_j \xrightarrow{\partial} C_{j-1}$$

$\partial\{v\} = 0$

$\partial(a) = v - v = 0 = \partial b = \partial c$

$\partial A_1 = c - b - a$, $\partial A_2 = a + b - c$

B_j = set of boundaries inside C_j $= \text{Image}(\partial|_{C_{j+1}})$

$B_0 = \{0\}$

$B_1 = \{\partial(z_1 A_1 + z_2 A_2) : z_1, z_2 \in \mathbb{Z}\} = \text{span}\{a+b-c\}$

$B_2 = \{0\}$

$\begin{pmatrix}\text{set of}\\ \text{cycles}\end{pmatrix} = Z_j = \ker(\partial|_{C_j}) =$

$Z_0 = C_0 = \text{span}\{v\}$

$Z_1 = C_1 = \text{span}\{a, b, c\}$

$Z_2 = \{z_1 A_1 + z_2 A_2 : \partial(\cdot) = 0\} = \text{span}\{A_1 + A_2\}$.

Notice: $\partial(\partial\, \cdot) = 0$

$\Rightarrow B_j < Z_j < C_j$ all these are abelian groups

We define: homology $H_j = \dfrac{Z_j}{B_j}$ an abelian group!

$H_0 = \dfrac{\text{span}\{v\}}{\{0\}} = \text{span}\{v\} \cong \mathbb{Z}$

$H_1 = \dfrac{\text{span}\{a,b,c\}}{\text{span}\{a+b-c\}} \cong \mathbb{Z}^2$ ($\text{span}\{a,b,c\} \cong \mathbb{Z}^3$, $\text{span}\{a+b-c\} \cong \mathbb{Z}$) can prove this.

$H_2 = \dfrac{\text{span}\{A_1 + A_2\}}{\{0\}} \cong \mathbb{Z}$.

shocking:

① Any subdivision yields same groups (up to isomorphism).

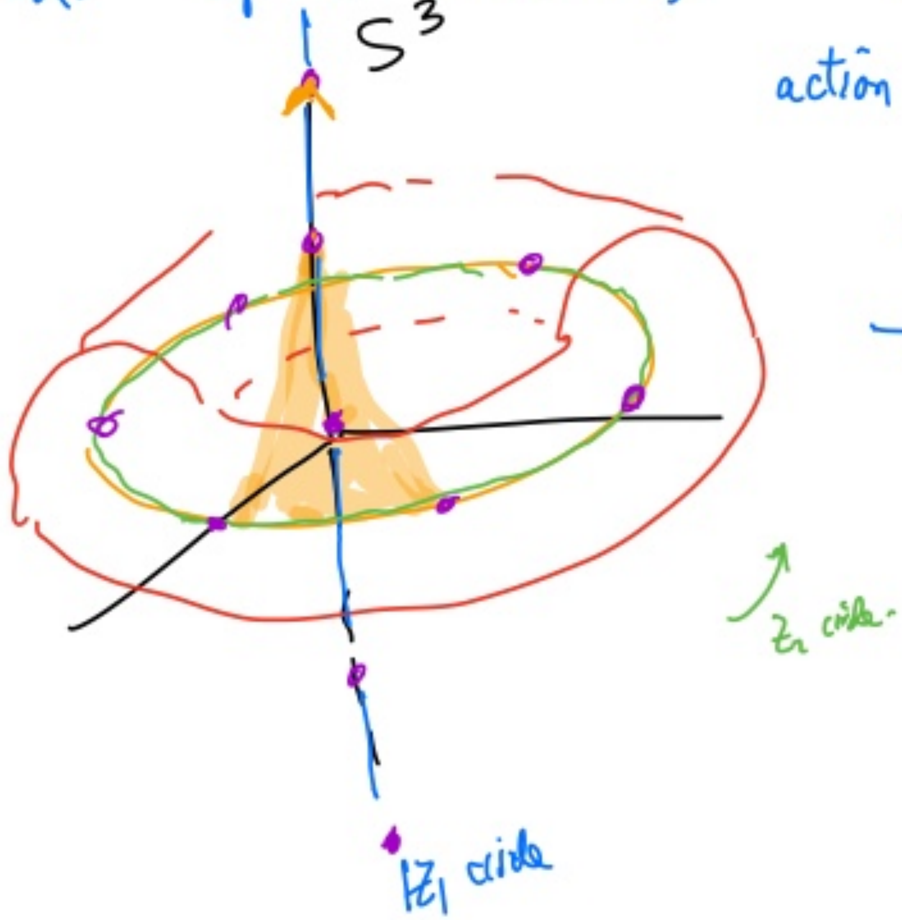
② Homotopy equivalent closed manifolds have same homology groups!

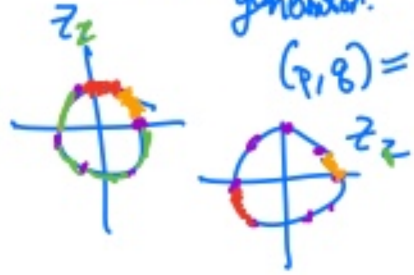
Lens space $L(p,q)$ $\qquad (z_1, z_2) \in S^3 \subseteq \mathbb{C}^2$

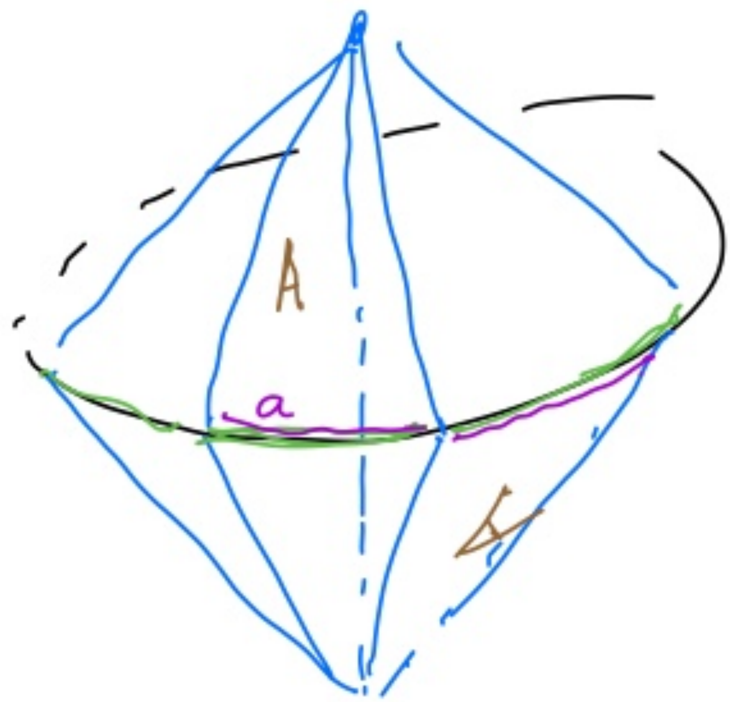
action $(z_1, z_2) \mapsto (e^{i\frac{2\pi}{p}} z_1, e^{i\frac{2\pi q}{p}} z_2)$

generator.

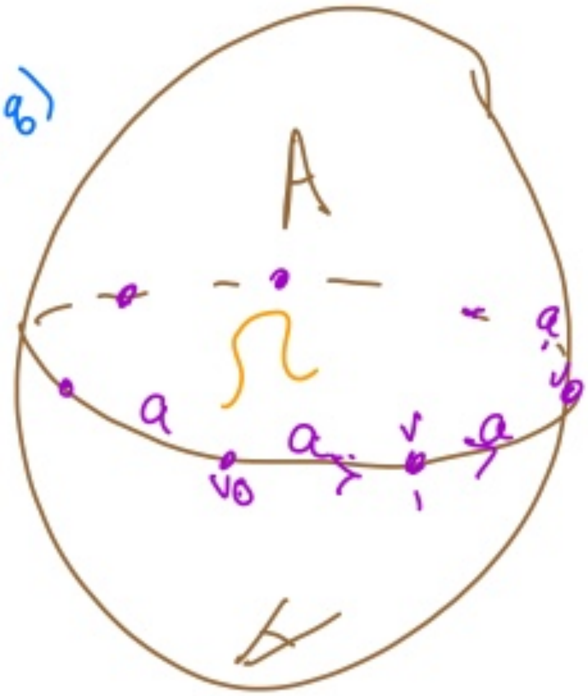
$(p,q) = 1$







$L(p,q)$



Cells
Chains

$C_3 = \{ n\Omega : n \in \mathbb{Z} \}$.

$C_2 = \{ nA : n \in \mathbb{Z} \}$

$C_1 = \{ na : n \in \mathbb{Z} \}$

$C_0 = \{ nv : n \in \mathbb{Z} \}$

$\partial(\Omega) = A - A = 0$

$\partial(A) = pa$

$\partial(a) = v - v = 0$

$\partial v = 0$

$B_3 = \{0\}$

$B_2 = \{0\}$

$B_1 = \{ npa : n \in \mathbb{Z} \}$

$B_0 = \{0\}$.

$Z_3 = \{ n\Omega : n \in \mathbb{Z} \} \Rightarrow H_3 = \frac{Z_3}{B_3} = \{n\Omega\} \cong \mathbb{Z}$

$Z_2 = \{0\}$

$Z_1 = \{ na : n \in \mathbb{Z} \}$.

$Z_0 = \{ nv : n \in \mathbb{Z} \}$

$H_2 = \frac{Z_2}{B_2} = \frac{\{0\}}{\{0\}} \cong \{0\}$

$H_1 = \frac{Z_1}{B_1} = \frac{\{na\}}{\{npa\}} \cong \frac{\mathbb{Z}}{p\mathbb{Z}} \cong \mathbb{Z}_p$

$H_0 = \frac{Z_0}{B_0} \cong \mathbb{Z}$

Twisted Homology:

multiply all cells & chains by $\beta^j \in \mathbb{C}$ $\quad 0 \le j \le p-1$, $\beta = e^{i\frac{2\pi}{p}}$

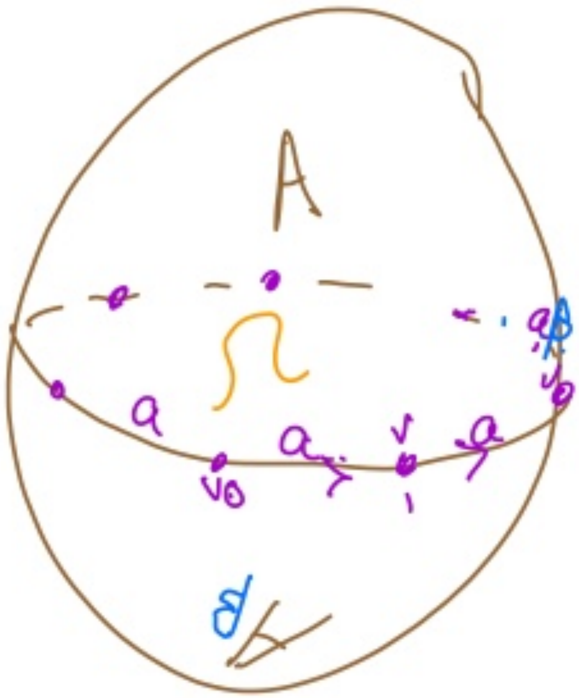
when we do identifications, multiply by β.

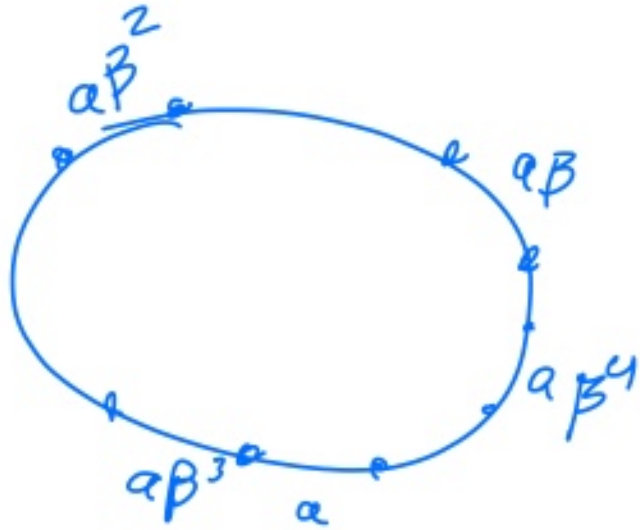
$C_3 = \{ n\Omega\beta^j : n \in \mathbb{Z}, j \in \mathbb{Z}_p \}$

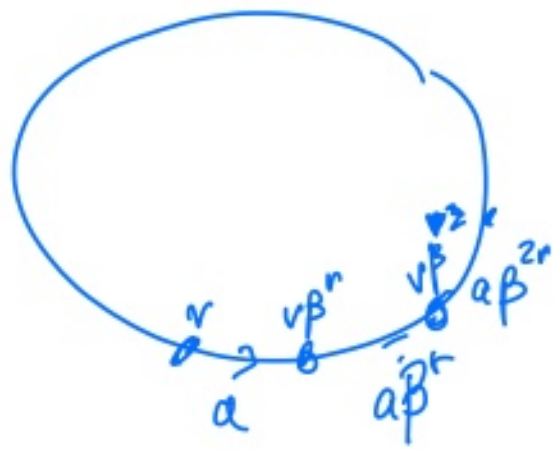
$C_2 = \{ nA\beta^j \dots \}$

$C_1 = \{ na\beta^j \dots \}$

$C_0 = \{ nv\beta^j \dots \}$







$\partial(\Omega) = A - A\beta$
$= (1-\beta)A$

$\partial(A) = a + a\beta^r + a\beta^{2r} + a\beta^{3r} \dots a\beta^{r(p-1)}$

$= a\left(\sum_{k=0}^{p-1} \beta^{rk}\right) = a\frac{(1-\beta^{rp})}{1-\beta^r} = 0$

$\beta^{rq} = \beta^1$

$rq \equiv 1 \mod p.$

$\partial(a) = (\beta^{r} - 1)v$

$\partial r = 0$

$\partial \Omega = (1-\beta)A$

$\partial A = 0$

$\partial a = (\beta^{r}-1)v$

$\partial r = 0$

$B_3 = \{0\}$

$B_2 =$

$H_j = \{0\} \ \forall j$

$C_j \cong B_j \oplus B_{j-1}$

Computing Torsion

change of basis.

$$\tau = \frac{[B_0/C_0][B_2\tilde{B}_1/C_2]}{[B_1\tilde{B}_0/C_1]}$$

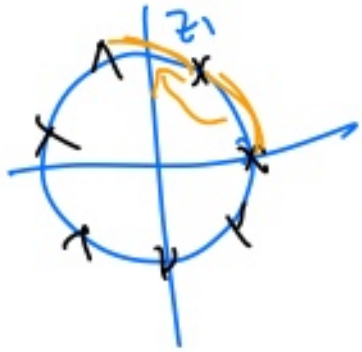
$$\tau = (\beta^{r}-1)(1-\beta)$$

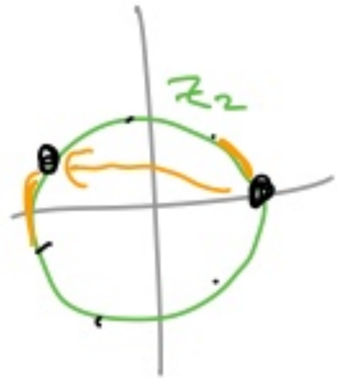
Last Talk

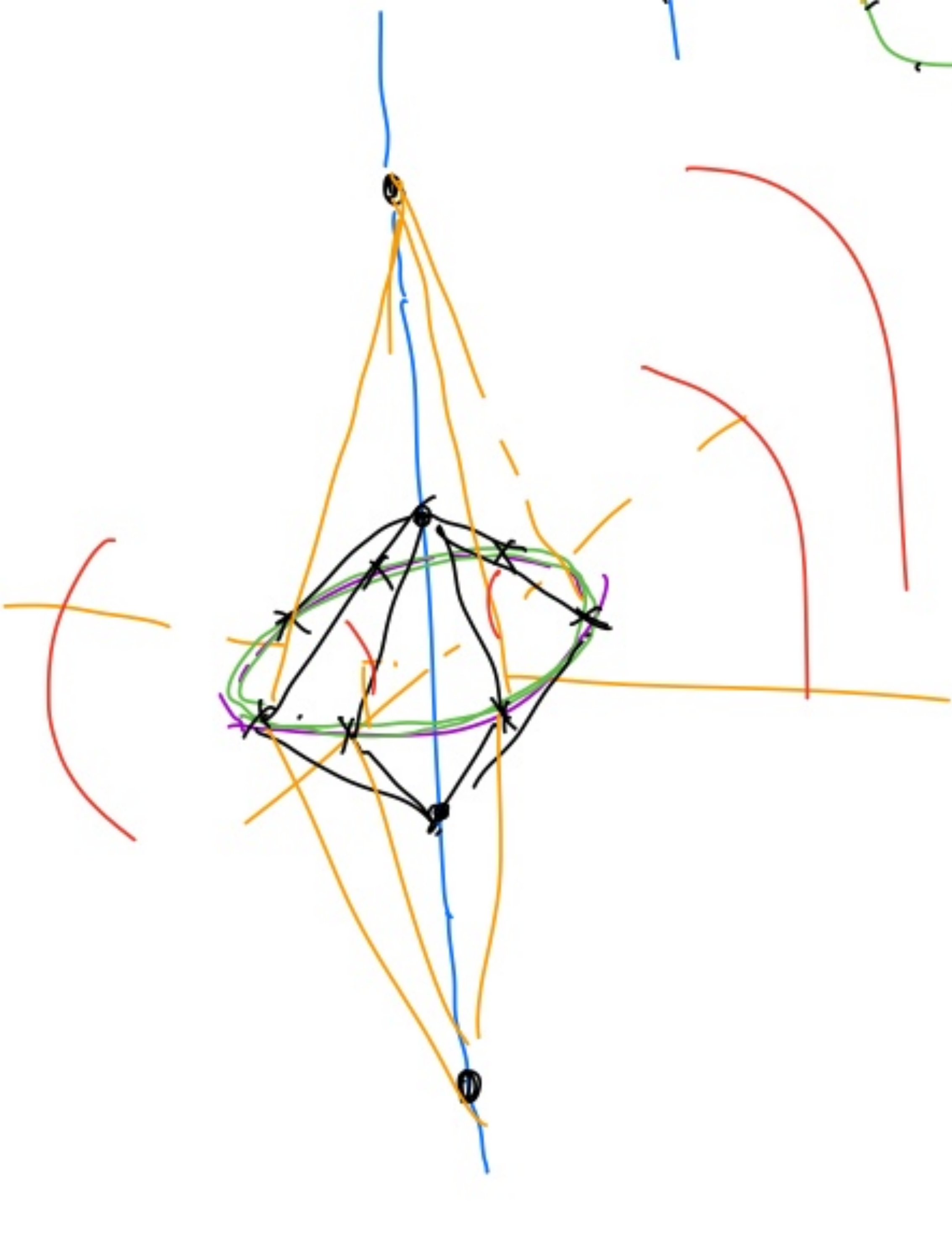
Review

S^3 with $\mathbb{Z}_p$ action $(z_1, z_2) \overset{\text{generator}}{\longmapsto} \left(e^{\frac{2\pi i}{p}} z_1, e^{\frac{2\pi i q}{p}} z_2\right)$

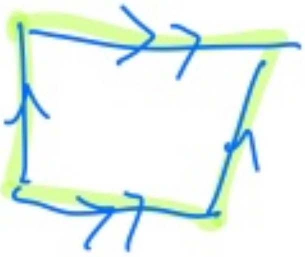
$S^3/\mathbb{Z}_p = L(p,q)$

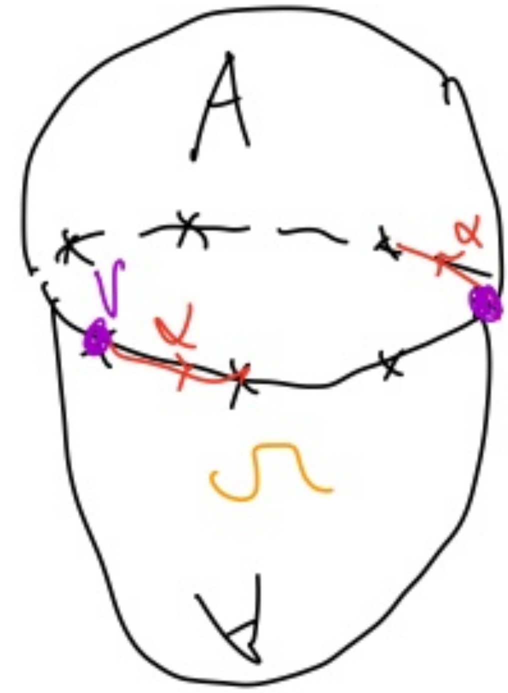






The lens





Ω = whole inside 3-d cell

A = top face = bottom face 2-d cell

α = edge = 1-d cell

v = vertex = 0-d cell

Homology

Chains

$C_3 = \text{span}\{\Omega\} = \{n\Omega : n \in \mathbb{Z}\} \cong \mathbb{Z}$

addition $3\Omega + 7\Omega = 10\Omega$

$C_2 = \text{span}\{A\} \cong \mathbb{Z}$

$C_1 = \text{span}\{\alpha\} \cong \mathbb{Z}$

$C_0 = \text{span}\{v\} \cong \mathbb{Z}.$

boundary maps

Boundaries $B_3 = \{0\}$

$\partial\Omega = A - A = 0 \Rightarrow B_2 = \{0\}$

$\partial A = p\alpha \Rightarrow B_1 = \{pn\alpha : n \in \mathbb{Z}\}$

$\partial\alpha = v - v = 0 \Rightarrow B_0 = \{0\}$

$\partial v = 0 \Rightarrow$

Cycles = chains x s.t. $\partial x = 0$

$Z_3 = \text{span}\{\Omega\}$

$Z_2 = \{0\}$

$Z_1 = \text{span}\{\alpha\}$

$Z_0 = \text{span}\{v\}$

$\Rightarrow$

Homology $H_j = \frac{Z_j}{B_j}$

$H_3 = \frac{Z_3}{B_3} = \frac{\text{span}\{\Omega\}}{\{0\}} \cong \mathbb{Z}$

$H_2 = \frac{Z_2}{B_2} = \frac{\{0\}}{\{0\}} \cong \{0\}$

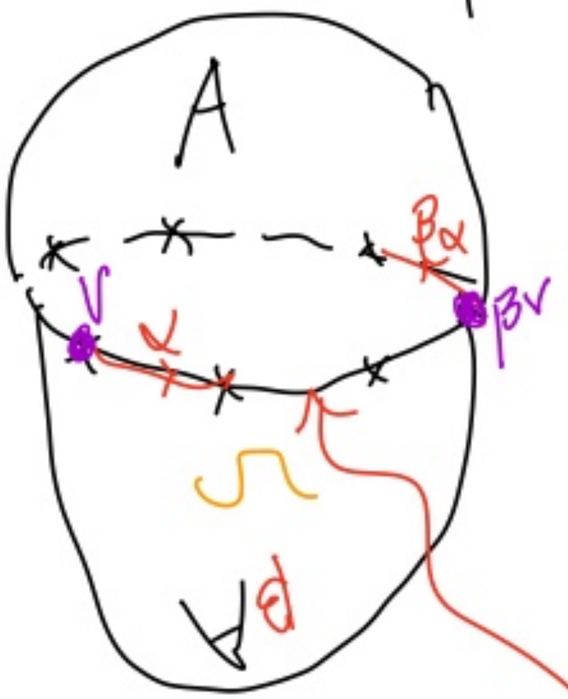
$Z_1 = \frac{Z_1}{B_1} = \frac{\text{span}\{\alpha\}}{\text{span}\{p\alpha\}} \cong \frac{\mathbb{Z}}{p\mathbb{Z}} \cong \mathbb{Z}_p$

$Z_0 = \frac{Z_0}{B_0} = \frac{\text{span}\{v\}}{\{0\}} \cong \mathbb{Z}.$

Twisted Homology $\beta = e^{i2\pi/p}$ $\beta^p = 1$

every action by generator of $\mathbb{Z}_p$ on S^3 results in multiplication by β.

Chains will have coefficients in $\mathbb{Z}\otimes\beta^j$ eg $3\cdot\beta^4$.



Note this edge would be $\beta^n\alpha$, where $\left(e^{i\frac{2\pi q}{p}}\right)^n = e^{i\frac{2\pi}{p}}$, ie n such that $nq \equiv 1 \mod p$.

next one would be $\beta^{2n}\alpha$

Twisted Chains

$\tilde{C}_3 =_{sp} \{n\beta^k\Omega : n,k\in\mathbb{Z}\} \cong \mathbb{Z}\times\mathbb{Z}_p$

$\tilde{C}_2 =_{sp} \{n\beta^k A\} \cong \mathbb{Z}\times\mathbb{Z}_p$

$\tilde{C}_1 =_{sp} \{n\beta^k\alpha\}$

$\tilde{C}_0 =_{sp} \{n\beta^k v\}$

Boundary maps

$\partial\Omega = A - \beta A = (1-\beta)A$

$\partial A = \alpha + \beta^n\alpha + \beta^{2n}\alpha + \dots + \beta^{(p-1)n}\alpha = \left(\sum_{i=0}^{p-1}\beta^{in}\right)\alpha = \frac{\beta^p-1}{\beta-1}\alpha = 0$

$\partial\alpha = \beta^n v - v = (\beta^n-1)v$

$\partial v = 0$

Boundaries are

$B_3 = \{0\}$

$B_2 =_{sp} \{(1-\beta) n \beta^k A : n, k \in \mathbb{Z}\}$

$B_1 = \{0\}$

$B_0 =_{sp} \{(\beta^{n}-1) n \beta^k v : n, k \in \mathbb{Z}\}$

Cycles are

$Z_3 = \{0\}$

$Z_2 =_{sp} \{n \beta^k A : n, k \in \mathbb{Z}\}$

$Z_1 = \{0\}$

$Z_0 =_{sp} \{n \beta^k v : n, k \in \mathbb{Z}\}$

$$\Rightarrow H_j = \frac{Z_j}{B_j} \cong \{0\} \ \forall j.$$

<u>R-torsion:</u> Choose natural basis (wrt coefficients $n\beta^k$) of $B_j, C_j \ \forall j$. Assume C_j basis is orthonormal. (have to prove this)

Since $H_j = 0 \ \forall j$, $C_j \cong B_j \oplus B_{j-1}$

$\cong B_j \oplus \widetilde{B}_{j-1}$ ← stuff in C_j that gets mapped to B_{j-1}, $\perp$ to B_j.

This gives a natural basis.

This gives a natural basis

Let $[B_j \widetilde{B}_{j-1} / C_j]$ = det of change of basis matrix going from $B_j \widetilde{B}_{j-1}$ to C_j, ie the determinant of the matrix whose columns are the vector basis of $B_j \widetilde{B}_{j-1}$.

$$\tau = \frac{[B_0/C_0][B_2 \widetilde{B}_1 / C_2] \cdots}{[B_1 \widetilde{B}_1 / C_1]\,[B_3 \widetilde{B}_3 / C_3] \cdots}$$

In our example, these are all 1x1 matrices Ha!

$$\partial \Omega = A - \beta A = (1-\beta)A$$

$$\partial A = \alpha + \beta^{r}\alpha + \beta^{2r}\alpha + \dots + \beta^{(p-1)r}\alpha = \left(\sum_{i=0}^{p-1}\beta^{ir}\right)\alpha = \frac{\beta^{p}-1}{\beta-1}\alpha = 0$$

$$\partial \alpha = \beta^{r} v - v = (\beta^{r}-1)v$$

$$\partial v = 0$$

$[B_0/C_0] = (\beta^{r}-1)$ ⟵ from $(\beta^{r}-1)v$ vs v

$[B_1\tilde{B}_0/C_1] = 1$ ⟵ from α vs. α

B_1: zippo

$[B_2\tilde{B}_1/C_2] = (1-\beta)$ ⟵ from $(1-\beta)A$ vs. A

$\tilde{B}_1$: zippo

$[B_3\tilde{B}_2/C_3] = 1$ ⟵ from Ω vs Ω

B_3: zippo, $\tilde{B}_2$: Ω

$$\Longrightarrow \tau = \frac{[B_0/C_0]\,[B_2\tilde{B}_1/C_2]}{[B_1\tilde{B}_0/C_1]\,[B_3\tilde{B}_2/C_3]} = \boxed{(\beta^{r}-1)(1-\beta)}$$

We could have made different choices ⇝

τ could be $\pm(\beta^{dr}-1)(1-\beta^{d})\beta^{s}$ for some s, d.

Using Number Theory, $\pm(1-\beta)(\beta^{r}-1) = (\beta^{dr'}-1)(1-\beta^{d})\beta^{s}$ implies $r = \pm r' \bmod p \Rightarrow \underbrace{q = \pm q' \bmod p \text{ or } qq' = \pm 1 \bmod p}$

Thm $L(p,q) \cong L(p',q') \Leftrightarrow p = p'$ and that

⇝ $L(7,1) \simeq L(7,2)$ but $L(7,1) \not\cong L(7,2)$.

Alternate Calculation of τ

Use orthonormal basis of $\tilde{C}_j$ to make inner product on each C_j

$$0 \to \tilde{C}_3 \underset{\partial^*}{\overset{\partial}{\rightleftarrows}} \tilde{C}_2 \underset{\partial^*}{\overset{\partial}{\rightleftarrows}} \tilde{C}_1 \underset{\partial^*}{\overset{\partial}{\rightleftarrows}} \tilde{C}_0 \xrightarrow{\partial} 0$$

adjoint of ∂ wrt inner products on C_3, C_2

recall $T : V \to W$ linear

$$\langle Ta, b \rangle_W = \langle a, T^* b \rangle_V$$

$a \in V,\ b \in W$

$T^* = \overline{T^T}$ if it's a matrix.

Combinatorial Laplacian

$$\Delta_0 = \partial\partial^* + \partial^*\partial \big|_{\tilde{C}_0}$$

$$\Delta_1 = \partial\partial^* + \partial^*\partial \big|_{\tilde{C}_1}$$

Δ_2

Δ_3

$$\Rightarrow \tau = \sqrt{\frac{(\det \Delta_1)^1 (\det \Delta_3)^3}{(\det \Delta_2)^2 \cdots}}$$

$$
\left.
\begin{aligned}
\partial \Omega &= A - \beta A = (1-\beta)A \\
\partial A &= \alpha + \beta^{r}\alpha + \beta^{2r}\alpha + \dots + \beta^{(p-1)r}\alpha = \Big(\sum_{j=0}^{r} \beta^{jr}\Big)\alpha = \frac{\beta^{p}-1}{\beta-1}\alpha = 0 \\
\partial \alpha &= \beta^{r} v - v = (\beta^{r}-1)v \\
\partial v &= 0
\end{aligned}
\right\}
\Rightarrow
\begin{aligned}
\partial^{*}\Omega &= 0 \\
\partial^{*}A &= (1-\beta)\Omega \\
\partial^{*}\alpha &= 0 \\
\partial^{*}v &= (\beta^{r}-1)\alpha
\end{aligned}
$$

Laplacians

$$
\begin{aligned}
\Delta \Omega &= (1-\beta)^{2}\Omega \\
\Delta A &= (1-\beta)^{2}A \\
\Delta \alpha &= (\beta^{r}-1)^{2}\alpha \\
\Delta v &= (\beta^{r}-1)^{2}v
\end{aligned}
$$

$$
\tau = \sqrt{\frac{(\det \Delta_1)^{1}(\det \Delta_3)^{3}}{(\det \Delta_2)^{2}\cdots}} = \sqrt{\frac{(\beta^{r}-1)^{2}\big((1-\beta)^{2}\big)^{3}}{\big((1-\beta)^{2}\big)^{2}}}
$$

$$
= \pm(\beta^{r}-1)(1-\beta)
$$

$L(p,q)$

$(7,1)$ $r=1$ $1\cdot 1 \equiv 1 \bmod 7$

$(7,2)$ $2\cdot 4 \equiv 1 \bmod 7$

$$t = \pm(\beta^{1}-1)(1-\beta) \qquad \beta = e^{i2\pi/p}$$

$$t = \pm(\beta^{4}-1)(1-\beta)$$